

Data compression of dynamic set-valued information systems

Guangming Lang^a Qingguo Li^{a *}

^a College of Mathematics and Econometrics, Hunan University
Changsha, Hunan 410082, P.R. China

Abstract. This paper further investigates the set-valued information system. First, we bring forward three tolerance relations for set-valued information systems and explore their basic properties in detail. Then the data compression is investigated for attribute reductions of set-valued information systems. Afterwards, we discuss the data compression of dynamic set-valued information systems by utilizing the precious compression of the original systems. Several illustrative examples are employed to show that attribute reductions of set-valued information systems can be simplified significantly by our proposed approach.

Keywords: Rough set; Set-valued information system; Attribute reduction; Homomorphism; Data compression

1 Introduction

Rough set theory, as a powerful mathematical tool to deal with vagueness and uncertainty of information, was proposed by Pawlak [26–29] in the early 1980s. But the requirement of the equivalence relation limits the applications of rough sets in many practical situations. To apply rough set theory to more complex data sets, it has been extended by combining with fuzzy sets [1–6, 10, 12, 16, 17, 24, 25], probability theory [8, 31, 32, 37–39, 45], topology [9, 11, 35, 36, 40, 42] and matroid theory [34].

Originally, the theory of rough sets based data analysis starts from the single-valued information system. In practice, it may often happen that some of attribute values for an object are set-valued. Recently, the set-valued information system has become a rapidly developing research area and got a lot of attention. For example, Guan et al. [15] initially introduced the set-valued information system as generalized models of single-valued information systems. Then Qian et al. [30] studied the set-valued ordered information system. Afterwards, many researchers [7, 19, 20, 22, 23, 41] investigated the dynamic set-valued information system. In the literature [15], the tolerance relation which discerns objects on the basis of that whether there exists common attribute values or not neglects some other difference. For example, it

*Corresponding author. Tel./fax: +86 731 8822855, liqingguoli@yahoo.com.cn(G. Q. Li)
E-mail address: langguangming1984@126.com(G. M. Lang)

may happen that there are two (respectively, ten) common values between objects A and B (respectively, A and C) with respect to an attribute, and objects B and C belong to the same tolerance class of object A. Although the number of common attribute values between objects A and B is larger than that between objects A and C, the tolerance relation cannot discern objects B and C in the tolerance class of object A. Therefore, it is of interest to introduce some tolerance relations for solving the above issue.

Meanwhile, homomorphisms [13, 14, 18, 22, 33, 43, 44] have been considered as an important approach for attribute reductions of information systems. For instance, Grzymala-Busse [14] initially introduced seven kinds of homomorphisms of knowledge representation systems and investigated their basic properties in detail. Then Li et al. [18] investigated invariant characters of information systems under some homomorphisms. Afterwards, many scholars [13, 33, 43, 44] discussed the relationship between information systems by means of homomorphisms. In practical situations, there exist a great many set-valued information systems. Inspired by the above work, attribute reductions of set-valued information systems may be conducted by means of homomorphisms. But so far few attempts have been made on the data compression of set-valued information systems under the condition of homomorphisms. In addition, the information system varies with time due to the dynamic characteristics of data collection, and the non-incremental approach to compressing the dynamic set-valued information system is often very costly or even intractable. Therefore, it is interesting to apply an incremental updating scheme to maintain the compression dynamically and avoid unnecessary computations by utilizing the compression of the original set-valued information system.

The purpose of this paper is to study the set-valued information system further. First, we introduce three tolerance relations for the set-valued information system and investigate their basic properties. Subsequently, the discernibility matrix based on the proposed relation is presented for attribute reductions of set-valued information systems. Second, we discuss the data compression of set-valued information systems. Concretely, a large-scale set-valued information system can be compressed into a relative-small relation information system under the condition of a homomorphism, and their attribute reductions are equivalent to each other. Third, the data compression of dynamic set-valued information systems is investigated by utilizing the precious compression of the original information systems. There are four types of dynamic set-valued information systems: adding and deleting attributes, adding and deleting objects. Using the proposed approach, the time complexity for computing attribute reducts of set-valued information systems can be reduced greatly by avoiding unnecessary computations.

The rest of this paper is organized as follows: Section 2 briefly reviews the basic concepts of set-valued information systems and consistent functions. In Section 3, we put forward three tolerance relations for the set-valued information system and investigate their basic properties in detail. We also present the discernibility matrix based on the proposed relation. Section 4 is devoted to discussing the data compression of set-valued information systems. In Section 5, we investigate the data compression of dynamic set-valued information systems. We conclude the paper in Section 6.

2 Preliminaries

In this section, we briefly review some concepts of the set-valued information system and the relation information system. In addition, an example is employed to illustrate the set-valued information system.

Definition 2.1 [15] Suppose $S = (U, A, V, f)$ is a set-valued information system, where $U = \{x_1, x_2, \dots, x_n\}$ is a non-empty finite set of objects, $A = \{a_1, a_2, \dots, a_m\}$ is a non-empty finite set of attributes, V is the set of attribute values, f is a mapping from $U \times A$ to V , where $f : U \times A \longrightarrow 2^V$ is a set-valued mapping.

It is obvious that the classical information system can be regarded as a special case of the set-valued information system. There are many semantic interpretations for the set-valued information system, we summarize two types of them as follows:

Type 1: For $x \in U, a \in A$, $f(x, a)$ is interpreted conjunctively. For example, if a is the attribute “speaking language”, then $f(x, a) = \{\text{German, French, Polish}\}$ can be viewed as: x speaks German, French and Polish, and x can speak three languages.

Type 2: For $x \in U, a \in A$, $f(x, a)$ is interpreted disjunctively. For instance, if a is the attribute “speaking language”, then $f(x, a) = \{\text{German, French, Polish}\}$ can be regarded as: x speaks German, French or Polish, and x can speak only one of them.

Definition 2.2 [15] Let $S = (U, A, V, f)$ be a set-valued information system, $a \in A$, and $B \subseteq A$. Then the tolerance relations R_a and R_B are defined as

$$\begin{aligned} R_a &= \{(x, y) | f(x, a) \cap f(y, a) \neq \emptyset, x, y \in U\}; \\ R_B &= \{(x, y) | \forall b \in B, f(x, b) \cap f(y, b) \neq \emptyset, x, y \in U\}. \end{aligned}$$

In other words, $(x, y) \in R_B$ is viewed as x and y are indiscernible with respect to B , and $R_B(x)$ is seen as the tolerance class for x with respect to B . Naturally, $R_B = \bigcap_{b \in B} R_b$. In spite of that the tolerance relation has been applied successfully in many fields, there exist some issues which need to be solved in practical situations. We employ an example to illustrate the problems of the tolerance relation presented in Definition 2.2 as below.

Example 2.3 Table 1 depicts a set-valued information system. In the sense of Definition 2.2, $R_{a_1}(x_2) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Obviously, we have that $(x_1, x_2), (x_3, x_2) \in R_{a_1}$. But $|f(x_1, a_1) \cap f(x_2, a_1)| = 1$ and $|f(x_2, a_1) \cap f(x_3, a_1)| = 2$. Furthermore, we obtain that $(x_1, x_4), (x_6, x_4) \in R_{a_1}$. But $f(x_1, a_1) \cap f(x_4, a_1) = \{0\}$ and $f(x_6, a_1) \cap f(x_4, a_1) = \{1\}$. Although there are some difference between objects which are in the same tolerance class, R_{a_1} cannot discern them.

To compress the relation information system, Wang et al. presented the concept of consistent functions as follows.

Table 1: A set-valued information system.

U	a_1	a_2	a_3	a_4
x_1	$\{0\}$	$\{0\}$	$\{1, 2\}$	$\{1, 2\}$
x_2	$\{0, 1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{0, 1, 2\}$
x_3	$\{1, 2\}$	$\{1\}$	$\{1\}$	$\{1, 2\}$
x_4	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$	$\{1, 2\}$
x_5	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1\}$
x_6	$\{1\}$	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$

Definition 2.4 [33] Let U_1 and U_2 be two universes, f a mapping from U_1 to U_2 , the relation R a mapping from $U \times U$ to $\{0, 1\}$, and $[x]_f = \{y \in U_1 | f(x) = f(y)\}$. For any $x, y \in U_1$, if $R(u, v) = R(s, t)$ for any two pairs $(u, v), (s, t) \in [x]_f \times [y]_f$, then f is said to be consistent with respect to R .

Especially, if the consistent function is a surjection, then it is a homomorphism between relation information systems. We can compress a large-scale information system into a relatively small-scale one under the condition of a homomorphism. It has been proved that attribute reductions of the original system and image system are equivalent to each other. Therefore, the consistent functions provide an approach to studying the data compression of relation information systems.

3 The tolerance relation based the discernibility matrix for set-valued information systems

In this section, we propose three tolerance relations to address the problem illustrated in Example 2.3. Then we present the concept of a discernibility matrix based on the proposed tolerance relation for attribute reductions of set-valued information systems.

Definition 3.1 Let (U, A, V, f) be a set-valued information system, $a \in A$, and $B \subseteq A$. Then the tolerance relations $R_a^{\geq h}$ and $R_B^{\geq H_B}$ are defined as

$$\begin{aligned}
 R_a^{\geq h} &= \{(x, y) | |f(x, a) \cap f(y, a)| \geq h, x, y \in U\}; \\
 R_B^{\geq H_B} &= \{(x, y) | |f(x, a_i) \cap f(y, a_i)| \geq h_i, x, y \in U, a_i \in B\},
 \end{aligned}$$

where $|\cdot|$ denotes the cardinality of a set, $H_B = (h_1, h_2, \dots, h_m)$ and $h_i = 0$ if $a_i \notin B$.

From Definition 3.1, we see that the number of common attribute values between objects are considered in the tolerance relations. Furthermore, we obtain that $R_a = R_a^{\geq 1}$, $R_B^{\geq (1, 1, \dots, 1)} = R_B$ and $R_B^{\geq H_B} = \bigcap_{a_i \in B} R_{a_i}^{\geq h_i}$. For the convenient representation, we denote $R_B^{\geq H_B}(x) = [x]_B^{\geq H_B} = \{y | (x, y) \in R_B^{\geq H_B}\}$ in the following. We define that $K = (k_1, k_2, \dots, k_m) \leq H_B$ if and only if $k_i \leq h_i$ for $1 \leq i \leq m$. Specially, if $\{R_a^{\geq h}(x) | x \in U\}$ is a covering of U , then $R_a^{\geq h}$ is called the $\geq h$ -relation. In general, $R_a^{\geq h}$ and $R_B^{\geq H_B}$ are

symmetric and intransitive, $R_a^{\geq h}$ and $R_B^{\geq H_B}$ are not reflexive necessarily if $h > 1$ and $H_B \neq (1, 1, \dots, 1)$, respectively. For example, consider Table 1, we obtain that $R_{a_1}^{\geq 2}(x_1) = \emptyset$. That is, $(x_1, x_1) \notin R_{a_1}^{\geq 2}$.

Proposition 3.2 *Let (U, A, V, f) be a set-valued information system, and $B, C \subseteq A$. Then we have*

- (1) if $H_B \leq H_C \leq H_A$, then $R_A^{\geq H_A} \subseteq R_C^{\geq H_C} \subseteq R_B^{\geq H_B}$;
- (2) if $H_B \leq H_C \leq H_A$, then $[x]_A^{\geq H_A} \subseteq [x]_C^{\geq H_C} \subseteq [x]_B^{\geq H_B}$.

We notice that $[y]_B^{\geq H_B} \subseteq [x]_B^{\geq H_B}$ does not hold necessarily if $y \in [x]_B^{\geq H_B}$, and that $[y]_B^{\geq H_B} = [x]_B^{\geq H_B}$ does not imply $x = y$, which can be illustrated by the following example.

Example 3.3 *Consider Table 1, we obtain that $[x_1]_{\{a_1\}}^{\geq (1,0,0,0)} = \{x_1, x_2, x_4\}$. It is clear that $x_2 \in [x_1]_{\{a_1\}}^{\geq (1,0,0,0)}$ and $[x_2]_{\{a_1\}}^{\geq (1,0,0,0)} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Moreover, we have that $[x_4]_{\{a_1\}}^{\geq (1,0,0,0)} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Thus $[x_2]_{\{a_1\}}^{\geq (1,0,0,0)} = [x_4]_{\{a_1\}}^{\geq (1,0,0,0)}$. But $x_2 \neq x_4$.*

Definition 3.4 *Let $S = (U, A, V, f)$ be a set-valued information system, $\mathcal{R}_A^{\geq} = \{R_{a_1}^{\geq h_1}, R_{a_2}^{\geq h_2}, \dots, R_{a_m}^{\geq h_m}\}$, and $R_{a_i}^{\geq h_i}$ the $\geq h_i$ -relation. Then $(U, \mathcal{R}_A^{\geq})$ is called the induced \geq -relation information system of S .*

For the sake of convenience, we denote $R_{a_i}^{\geq h_i}$ as R_i and consider the situation that $h_i = 1$ in the following. An example is employed to illustrate the induced \geq -relation information system.

Example 3.5 *Consider Table 1, we obtain the induced \geq -relation information system $(U, \mathcal{R}_A^{\geq})$ and $\mathcal{R}_A^{\geq} = \{R_i | 1 \leq i \leq 4\}$, where*

$$\begin{aligned}
R_1(x_1) &= \{x_1, x_2, x_4\}, R_1(x_2) = R_1(x_4) = \{x_1, x_2, x_3, x_4, x_5, x_6\}, R_1(x_3) = R_1(x_5) = R_1(x_6) = \{x_2, x_3, x_4, \\
&\quad x_5, x_6\}; \\
R_2(x_1) &= \{x_1, x_4\}, R_2(x_2) = R_2(x_5) = \{x_2, x_3, x_4, x_5, x_6\}, R_2(x_3) = R_2(x_6) = \{x_2, x_3, x_5, x_6\}, R_2(x_4) = \{x_1, \\
&\quad x_2, x_4, x_5\}; \\
R_3(x_1) &= R_3(x_2) = R_3(x_3) = R_3(x_4) = R_3(x_5) = R_3(x_6) = \{x_1, x_2, x_3, x_4, x_5, x_6\}; \\
R_4(x_1) &= R_4(x_2) = R_4(x_3) = R_4(x_4) = R_4(x_5) = R_4(x_6) = \{x_1, x_2, x_3, x_4, x_5, x_6\}.
\end{aligned}$$

Definition 3.6 *Let $S = (U, A, V, f)$ be a set-valued information system, $(U, \mathcal{R}_A^{\geq})$ the induced \geq -relation information system of S , and $P \subseteq A$. If $\bigcap \mathcal{R}_P^{\geq} = \bigcap \mathcal{R}_A^{\geq}$ and $\bigcap \mathcal{R}_{P^*}^{\geq} \neq \bigcap \mathcal{R}_A^{\geq}$ for any $\mathcal{R}_{P^*}^{\geq} \subsetneq \mathcal{R}_P^{\geq}$, then \mathcal{R}_P^{\geq} is called a reduct of $(U, \mathcal{R}_A^{\geq})$.*

By Definition 3.6, we see that the reduct is the minimal subset of attribute set, which preserves the relation \mathcal{R}_A^{\geq} . For instance, we get the reduct $P = \{R_2\}$ in the sense of Definition 3.6 for the relation information system presented in Example 3.5.

Now we introduce the discernibility matrix based on Definition 3.1 and investigate its basic properties.

Definition 3.7 Let $S = (U, A, V, f)$ be a set-valued information system. Then its discernibility matrix $M_A = (M(x, y))$ is a $|U| \times |U|$ matrix, the element $M(x, y)$ is defined by

$$M(x, y) = \{a \in A \mid (x, y) \notin R_a^{\geq h_a}, x, y \in U\},$$

where $R_a^{\geq h_a}$ is $a \geq h_a$ -relation.

That is, the physical meaning of the matrix element $M(x, y)$ is that objects x and y can be distinguished by any element of $M(x, y)$. If we obtain that $M(x, y) \neq \emptyset$, then objects x and y can be discerned. It is sufficient to consider only the lower triangle or the upper triangle of the matrix since the discernibility matrix M is symmetric.

Definition 3.8 Let $S = (U, A, V, f)$ be a set-valued information system, and $M = (M(x, y))$ the discernibility matrix of S . Then $\Delta = \bigwedge_{(x, y) \in U^2} \bigvee M(x, y)$ is called the discernibility function of S .

The expression $\bigvee M(x, y)$ denotes the disjunction of all attributes in $M(x, y)$, and the expression $\bigwedge \{\bigvee M(x, y)\}$ stands for the conjunction of all $\bigvee M(x, y)$. In addition, $\bigwedge B$ is a prime implicant of the discernibility function Δ if and only if B is a reduct of S .

Next, we propose another two concepts of tolerance relations and discuss their basic properties for set-valued information systems.

Definition 3.9 Let (U, A, V, f) be a set-valued information system, $a \in A$, and $B \subseteq A$. Then the tolerance relations R_a^h and $R_B^{H_B}$ are defined as

$$\begin{aligned} R_a^h &= \{(x, y) \mid |f(x, a) \cap f(y, a)| = h, x, y \in U\}; \\ R_B^{H_B} &= \{(x, y) \mid |f(x, a_i) \cap f(y, a_i)| = h_i, x, y \in U, a_i \in B\}. \end{aligned}$$

From Definition 3.9, we see that R_a^h and $R_B^{H_B}$ are symmetric and intransitive, R_a^h and $R_B^{H_B}$ are not reflexive necessarily. Meanwhile, we have that $R_a^{\geq h} = \bigcup_{j \geq h} R_a^j$ and $R_B^{\geq H_B} = \bigcup_{K \geq H_B} R_B^K$. For the sake of simplicity, we note that $R_B^{H_B}(x) = [x]_B^{H_B} = \{y \mid (x, y) \in R_B^{H_B}\}$.

Property 3.10 Let (U, A, V, f) be a set-valued information system, and $B, C \subseteq A$. Then we have

- (1) if $H_B \leq H_C \leq H_A$, then $R_A^{H_A} \subseteq R_C^{H_C} \subseteq R_B^{H_B}$;
- (2) if $H_B \leq H_C \leq H_A$, then $[x]_A^{H_A} \subseteq [x]_C^{H_C} \subseteq [x]_B^{H_B}$.

Definition 3.11 Let (U, A, V, f) be a set-valued information system, $a \in A$, $B \subseteq A$, and $P \subseteq V_a$. Then the tolerance relations R_a^P and $R_B^{\mathcal{P}}$ are defined as

$$\begin{aligned} R_a^P &= \{(x, y) \mid f(x, a) \cap f(y, a) = P, x, y \in U\}; \\ R_B^{\mathcal{P}} &= \{(x, y) \mid f(x, a_i) \cap f(y, a_i) = P_i, x, y \in U, a_i \in B\}, \end{aligned}$$

where $\mathcal{P} = (P_1, P_2, \dots, P_m)$, and P_i is defined as $P_i \subseteq V_{a_i}$ (respectively, $P_i = \emptyset$) if $a_i \in B$ (respectively, $a_i \notin B$).

In the sense of Definitions 3.9 and 3.11, it is observed that $R_a^h = \bigcup \{R_a^P | P \in 2^A, |P| = h\}$. Furthermore, R_a^P and $R_B^{\mathcal{P}}$ are symmetric and intransitive. By Definitions 3.1, 3.9 and 3.11, we obtain that

$$R_a^{\geq h} = \bigcup_{i \geq h} R_a^i = \bigcup_{i \geq h} \bigcup \{R^P | |P| = i, P \in 2^{V_a}\}$$

and

$$R_B^{\geq I} = \bigcap_{a \in B} \left\{ \bigcup_{i \geq h} R_a^i \right\} = \bigcap_{a \in B} \left\{ \bigcup_{i \geq h} \bigcup \{R^P | |P| = i, P \in 2^{V_a}\} \right\}.$$

In addition, we can define discernibility matrixes based on Definitions 3.9 and 3.11, respectively. For the sake of simplicity, we do not present them in this section.

4 Data compression of the set-valued information system

In this section, we investigate the data compression of the large-scale set-valued information system. Concretely, we derive the induced \geq -relation information system of the set-valued information system. Then the induced \geq -relation information system is compressed into a relatively small one under the condition of a homomorphism, and attribute reductions of the original system and image system are equivalent to each other. In addition, we illustrate that the time complexity of computing attribute reductions can be reduced greatly by means of the compression from another view.

Definition 4.1 Let $(U_1, \mathcal{R}_A^{\geq})$ be the induced \geq -relation information system of the set-valued information system $S = (U_1, A, V, f)$, $R \in \mathcal{R}_A^{\geq}$, $[x]_R = \{y | R(x) = R(y), x, y \in U_1\}$, and $U_1/R = \{[x]_R | x \in U_1\}$. Then U_1/R is called the partition based on R .

Following, we employ Table 2 to show the partition based on each relation for the induced \geq -relation information system $(U_1, \mathcal{R}_A^{\geq})$, where P_{ix_j} stands for the block containing x_j in the partition based on the relation R_i . It is easy to see that $P_{Ax_j} = \bigcap_{1 \leq i \leq m} P_{ix_j}$, where P_{Ax_j} denotes the block containing x_j in the partition based on \mathcal{R}_A^{\geq} .

We present the algorithm of compressing the set-valued information system as follows.

Algorithm 4.2 Let $S = (U_1, A, V, f)$ be a set-valued information system, where $U_1 = \{x_1, \dots, x_n\}$ and $A = \{a_1, \dots, a_m\}$.

Step 1. Input the set-valued information system $S = (U_1, A, V, f)$ and obtain the induced \geq -relation information system $(U_1, \mathcal{R}_A^{\geq})$, where $\mathcal{R}_A^{\geq} = \{R_1, R_2, \dots, R_m\}$;

Step 2. Compute the partition U_1/R_i ($1 \leq i \leq m$) and obtain $U_1/\mathcal{R}_A^{\geq} = \{C_i | 1 \leq i \leq N\}$;

Table 2: The partitions based on each relation R_i ($1 \leq i \leq m$) and \mathcal{R}_A^\geq , respectively.

U_1	R_1	R_2	.	.	.	R_m	\mathcal{R}_A^\geq
x_1	P_{1x_1}	P_{2x_1}	.	.	.	P_{mx_1}	P_{Ax_1}
x_2	P_{1x_2}	P_{2x_2}	.	.	.	P_{mx_2}	P_{Ax_2}
.
.
.
x_n	P_{1x_n}	P_{2x_n}	.	.	.	P_{mx_n}	P_{Ax_n}

Step 3. Define the function $g(x) = y_i$ for any $x \in C_i$ and obtain $(U_2, g(\mathcal{R}_A^\geq))$, where $U_2 = \{g(x_i) | x_i \in U_1\}$ and $g(\mathcal{R}_A^\geq) = \{g(R_1), g(R_2), \dots, g(R_m)\}$;

Step 4. Obtain attribute reductions $\{g(R_{i1}), g(R_{i2}), \dots, g(R_{ik})\}$ of $(U_2, \{g(R_1), g(R_2), \dots, g(R_m)\})$;

Step 5. Obtain a reduct $\{R_{i1}, R_{i2}, \dots, R_{ik}\}$ of $(U_1, \mathcal{R}_A^\geq)$ and output the results.

The mapping g presented in Algorithm 4.2 is a homomorphism from $(U_1, \mathcal{R}_A^\geq)$ to $(U_2, g(\mathcal{R}_A^\geq))$ in the sense of Definition 2.4, and attribute reductions of $(U_1, \mathcal{R}_A^\geq)$ and $(U_2, g(\mathcal{R}_A^\geq))$ are equivalent to each other under the condition of the homomorphism g .

Remark. In Example 3.1 [33], Wang et al. only obtained the partition U_1/\mathcal{R}_A^\geq . But we get U_1/\mathcal{R}_A^\geq by computing U_1/R_i for any $R_i \in \mathcal{R}_A^\geq$ in Algorithm 4.2. By using the proposed approach, the data compression of dynamic set-valued information systems can be conducted on the basis of that of the original set-valued information system, which is illustrated in Section 5.

We give an example to show the data compression of set-valued information systems with Algorithm 4.2.

Table 3: A set-valued information system.

U_1	a_1	a_2	a_3	a_4
x_1	$\{0\}$	$\{0\}$	$\{1, 2\}$	$\{1, 2\}$
x_2	$\{0, 1, 2\}$	$\{0, 1, 2\}$	$\{1, 2\}$	$\{0, 1, 2\}$
x_3	$\{1, 2\}$	$\{0, 1\}$	$\{1, 2\}$	$\{1, 2\}$
x_4	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$	$\{1\}$
x_5	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1\}$
x_6	$\{1\}$	$\{1, 2\}$	$\{0, 1\}$	$\{0, 1\}$
x_7	$\{0\}$	$\{0\}$	$\{1, 2\}$	$\{1, 2\}$
x_8	$\{1\}$	$\{1, 2\}$	$\{0, 1\}$	$\{0, 1\}$

Example 4.3 Table 3 depicts the set-valued information system $S_1 = (U_1, A, V, f)$. According to Definitions 3.1 and 3.4, we obtain the induced \geq -relation information system $(U_1, \mathcal{R}_A^\geq)$, and $\mathcal{R}_A^\geq = \{R_1, R_2, R_3, R_4\}$,

where

$$\begin{aligned}
R_1(x_1) &= R_1(x_7) = \{x_1, x_2, x_4, x_7\}, R_1(x_2) = R_1(x_4) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \\
R_1(x_3) &= R_1(x_5) = R_1(x_6) = R_1(x_8) = \{x_2, x_3, x_4, x_5, x_6, x_8\}; \\
R_2(x_1) &= R_1(x_7) = \{x_1, x_2, x_3, x_4, x_7\}, R_2(x_2) = R_2(x_3) = R_2(x_4) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \\
R_2(x_5) &= R_2(x_6) = R_2(x_8) = \{x_2, x_3, x_4, x_5, x_6, x_8\}; \\
R_3(x_1) &= R_3(x_2) = R_3(x_3) = R_3(x_4) = R_3(x_5) = R_3(x_6) = R_3(x_7) = R_3(x_8) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}; \\
R_4(x_1) &= R_4(x_2) = R_4(x_3) = R_4(x_4) = R_4(x_5) = R_4(x_6) = R_4(x_7) = R_4(x_8) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}.
\end{aligned}$$

For the sake of convenience, we present $\{R_i(x_j) | x_j \in U_1\}$ instead of R_i in this work. By Definition 4.1, we derive the partitions U_1/R_1 , U_1/R_2 , U_1/R_3 and U_1/R_4 shown in Table 4. Then, based on U_1/R_1 , U_1/R_2 , U_1/R_3 and U_1/R_4 , we get the partition $U_1/\mathcal{R}_A^\geq = \{\{x_1, x_7\}, \{x_2, x_4\}, \{x_3\}, \{x_5, x_6, x_8\}\}$ and define a mapping $g : U_1 \longrightarrow U_2$ as follows:

$$g(x_1) = g(x_7) = y_1, g(x_2) = g(x_4) = y_2, g(x_3) = y_3, g(x_5) = g(x_6) = g(x_8) = y_4.$$

Afterwards, we derive the compressed relation information system $(U_2, g(\mathcal{R}_A^\geq))$, where $U_2 = \{y_1, y_2, y_3, y_4\}$, $g(\mathcal{R}_A^\geq) = \{g(R_1), g(R_2), g(R_3), g(R_4)\}$, and

$$\begin{aligned}
g(R_1)(y_1) &= \{y_1, y_2\}, g(R_1)(y_2) = \{y_1, y_2, y_3, y_4\}, g(R_1)(y_3) = g(R_1)(y_4) = \{y_2, y_3, y_4\}; \\
g(R_2)(y_1) &= \{y_1, y_2, y_3\}, g(R_2)(y_2) = g(R_2)(y_3) = \{y_1, y_2, y_3, y_4\}, g(R_2)(y_4) = \{y_2, y_3, y_4\}; \\
g(R_3)(y_1) &= g(R_3)(y_2) = g(R_3)(y_3) = g(R_3)(y_4) = \{y_1, y_2, y_3, y_4\}; \\
g(R_4)(y_1) &= g(R_4)(y_2) = g(R_4)(y_3) = g(R_4)(y_4) = \{y_1, y_2, y_3, y_4\}.
\end{aligned}$$

Finally, we obtain the following results:

- (1) g is a homomorphism from $(U_1, \mathcal{R}_A^\geq)$ to $(U_2, g(\mathcal{R}_A^\geq))$;
- (2) $g(R_2)$, $g(R_3)$ and $g(R_4)$ are superfluous in $g_1(\mathcal{R}_A^\geq)$ if and only if R_2 , R_3 and R_4 are superfluous in \mathcal{R}_A^\geq ;
- (3) $\{g(R_1)\}$ is a reduct of $g(\mathcal{R}_A^\geq)$ if and only if $\{R_1\}$ is a reduct of \mathcal{R}_A^\geq .

From Example 4.3, we see that the image system $(U_2, g(\mathcal{R}_A^\geq))$ has the relatively smaller size than the original system $(U_1, \mathcal{R}_A^\geq)$, and their attribute reductions are equivalent to each other under the condition of a homomorphism.

To illustrate that the time complexity of computing attribute reductions is reduced greatly by means of homomorphisms from another view, we employ an example to show attribute reductions on the basis of the discernibility matrix in the following.

Table 4: The partitions based on R_1, R_2, R_3, R_4 and \mathcal{R}_A^\geq , respectively.

U_1	R_1	R_2	R_3	R_4	\mathcal{R}_A^\geq
x_1	$\{x_1, x_7\}$	$\{x_1, x_7\}$	U_1	U_1	$\{x_1, x_7\}$
x_2	$\{x_2, x_4\}$	$\{x_2, x_3, x_4\}$	U_1	U_1	$\{x_2, x_4\}$
x_3	$\{x_3, x_5, x_6, x_8\}$	$\{x_2, x_3, x_4\}$	U_1	U_1	$\{x_3\}$
x_4	$\{x_2, x_4\}$	$\{x_2, x_3, x_4\}$	U_1	U_1	$\{x_2, x_4\}$
x_5	$\{x_3, x_5, x_6, x_8\}$	$\{x_5, x_6, x_8\}$	U_1	U_1	$\{x_5, x_6, x_8\}$
x_6	$\{x_3, x_5, x_6, x_8\}$	$\{x_5, x_6, x_8\}$	U_1	U_1	$\{x_5, x_6, x_8\}$
x_7	$\{x_1, x_7\}$	$\{x_1, x_7\}$	U_1	U_1	$\{x_1, x_7\}$
x_8	$\{x_3, x_5, x_6, x_8\}$	$\{x_5, x_6, x_8\}$	U_1	U_1	$\{x_5, x_6, x_8\}$

Example 4.4 (Continuation of Example 4.3) Based on Definition 3.7, we obtain the discernibility matrixes D_1 and D_2 of $(U_1, \mathcal{R}_A^\geq)$ and $(U_2, g(\mathcal{R}_A^\geq))$, respectively.

$$D_1 = \begin{bmatrix} \emptyset & & & & & & \\ \{a_1\} & \emptyset & & & & & \\ \emptyset & \emptyset & \emptyset & & & & \\ \{a_1, a_2\} & \emptyset & \emptyset & \emptyset & & & \\ \{a_1, a_2\} & \emptyset & \emptyset & \emptyset & \emptyset & & \\ \emptyset & \emptyset & \{a_1\} & \emptyset & \{a_1, a_2\} & \{a_1, a_2\} & \\ \{a_1, a_2\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{a_1, a_2\} \end{bmatrix},$$

and

$$D_2 = \begin{bmatrix} \emptyset & & \\ \{a_1\} & \emptyset & \\ \{a_1, a_2\} & \emptyset & \emptyset \end{bmatrix}.$$

It is obvious that the size of D_1 is larger than that of D_2 , and $\{a_1\}$ is the reduct of $(U_1, \mathcal{R}_A^\geq)$ and $(U_2, g(\mathcal{R}_A^\geq))$. We see that the time complexity of computing D_2 is relatively lower than that of computing D_1 .

From the practical viewpoint, it may be difficult to construct attribute reducts of a large-scale set-valued information system directly. However, we can convert it into a relation information system and compress the relation information system into a relatively smaller one under the condition of a homomorphism. Then we conduct the attribute reductions of the image system which is equivalent to that of the original information system. Therefore, the homomorphisms may provide a more efficient approach to dealing with attribute reductions of large-scale set-valued information systems.

5 Data compression of the dynamic set-valued information system

In this section, we consider the data compression of four types of dynamic set-valued information systems in terms of variations of the attribute and object sets.

5.1 Compressing the dynamic set-valued information system when adding an attribute set

Suppose $S_1 = (U_1, A, V_1, f_1)$ is a set-valued information system. By adding an attribute set P into A satisfying $A \cap P = \emptyset$, where $P = \{a_{m+1}, a_{m+2}, \dots, a_k\}$, we get the updated set-valued information system $S_2 = (U_1, A \cup P, V_2, f_2)$. There are three steps to compress S_2 by utilizing the compression of the original system S_1 . First, we obtain the induced \geq -relation information system $(U_1, \mathcal{R}_P^\geq)$ and derive the partition U_1/R_i based on $R_i \in \mathcal{R}_P^\geq$ ($m+1 \leq i \leq k$). Second, we get Table 5 by adding the partition U_1/R_i ($m+1 \leq i \leq k$) into Table 2 and derive the partition $U_1/\mathcal{R}_{A \cup P}^\geq$. Third, as Example 4.3, we define the homomorphism g based on $U_1/\mathcal{R}_{A \cup P}^\geq$ and derive the relation information system $S_3 = (g(U_1), g(\mathcal{R}_{A \cup P}^\geq))$.

Table 5: The partitions based on each relation R_i ($1 \leq i \leq k$) and $\mathcal{R}_{A \cup P}^\geq$, respectively.

U_1	R_1	R_2	.	.	.	R_k	$\mathcal{R}_{A \cup P}^\geq$
x_1	P_{1x_1}	P_{2x_1}	.	.	.	P_{kx_1}	$P_{(A \cup P)x_1}$
x_2	P_{1x_2}	P_{2x_2}	.	.	.	P_{kx_2}	$P_{(A \cup P)x_2}$
.
.
.
x_n	P_{1x_n}	P_{2x_n}	.	.	.	P_{kx_n}	$P_{(A \cup P)x_n}$

The following example is employed to illustrate the data compression of dynamic set-valued information systems when adding an attribute set.

Table 6: A set-valued information system by adding an attribute a_5 into Table 2.

U_1	a_1	a_2	a_3	a_4	a_5
x_1	$\{0\}$	$\{0\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$
x_2	$\{0, 1, 2\}$	$\{0, 1, 2\}$	$\{1, 2\}$	$\{0, 1, 2\}$	$\{0, 2\}$
x_3	$\{1, 2\}$	$\{0, 1\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$
x_4	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$	$\{1\}$	$\{2\}$
x_5	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1\}$	$\{2\}$
x_6	$\{1\}$	$\{1, 2\}$	$\{0, 1\}$	$\{0, 1\}$	$\{0, 1, 2\}$
x_7	$\{0\}$	$\{0\}$	$\{1, 2\}$	$\{1, 2\}$	$\{0, 2\}$
x_8	$\{1\}$	$\{1, 2\}$	$\{0, 1\}$	$\{0, 1\}$	$\{3\}$

Example 5.1 We obtain the updated set-valued information system shown in Table 6 by adding an attribute a_5 into the set-valued information system shown in Table 2. By Definition 4.1, we first get that $U_1/R_5 = \{\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}, \{x_8\}\}$ based on a_5 . Then we obtain Table 7 and derive $U_1/\mathcal{R}_{A \cup \{a_5\}}^\geq = \{\{x_1, x_7\}, \{x_2, x_4\}, \{x_3\}, \{x_5, x_6\}, \{x_8\}\}$. Afterwards, we define the mapping $g : U_1 \rightarrow U_2$ as follows:

$$g(x_1) = g(x_7) = y_1, g(x_2) = g(x_4) = y_2, g(x_3) = y_3, g(x_5) = g(x_6) = y_4, g(x_8) = y_5,$$

where $U_2 = \{y_1, y_2, y_3, y_4, y_5\}$. Consequently, we obtain the relation information system $(U_2, g(\mathcal{R}_{A \cup \{a_5\}}^\geq))$. For simplicity, we do not list the relation information system in this subsection.

Table 7: The partitions based on R_1, R_2, R_3, R_4, R_5 and $\mathcal{R}_{A \cup \{a_5\}}^\geq$, respectively.

U_1	R_1	R_2	R_3	R_4	R_5	$\mathcal{R}_{A \cup \{a_5\}}^\geq$
x_1	$\{x_1, x_7\}$	$\{x_1, x_7\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$	$\{x_1, x_7\}$
x_2	$\{x_2, x_4\}$	$\{x_2, x_3, x_4\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$	$\{x_2, x_4\}$
x_3	$\{x_3, x_5, x_6, x_8\}$	$\{x_2, x_3, x_4\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$	$\{x_3\}$
x_4	$\{x_2, x_4\}$	$\{x_2, x_3, x_4\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$	$\{x_2, x_4\}$
x_5	$\{x_3, x_5, x_6, x_8\}$	$\{x_5, x_6, x_8\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$	$\{x_5, x_6\}$
x_6	$\{x_3, x_5, x_6, x_8\}$	$\{x_5, x_6, x_8\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$	$\{x_5, x_6\}$
x_7	$\{x_1, x_7\}$	$\{x_1, x_7\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$	$\{x_1, x_7\}$
x_8	$\{x_3, x_5, x_6, x_8\}$	$\{x_5, x_6, x_8\}$	$\{U_1\}$	$\{U_1\}$	$\{x_8\}$	$\{x_8\}$

In Example 5.1, we compress the dynamic set-valued information system when adding an attribute. The same approach can be applied to the dynamic set-valued information system when adding an attribute set.

5.2 Compressing the dynamic set-valued information system when deleting an attribute set

Suppose $S_1 = (U_1, A, V_1, f_1)$ is a set-valued information system. By deleting an attribute $a_l \in A$, we get the updated set-valued information system $S_2 = (U_1, A - \{a_l\}, V_2, f_2)$. First, we obtain Table 8 by deleting the partition U_1/R_l shown in Table 2. Second, we get the partition $U/\mathcal{R}_{(A - \{a_l\})}^\geq$ based on U_1/R_i ($1 \leq i \leq l - 1, l + 1 \leq i \leq m$) and define the homomorphism g as Example 4.3. Third, we obtain the relation information system $S_3 = (g(U_1), g(\mathcal{R}_{(A - \{a_l\})}^\geq))$. We can compress the dynamic set-valued information system when deleting an attribute set with the same approach.

We employ an example to illustrate that how to compress the dynamic set-valued information system when deleting an attribute set as follows.

Example 5.2 By deleting the attribute a_1 in the set-valued information system S_1 shown in Table 3, we obtain the updated set-valued information system S_2 shown in Table 9. To compress the updated

Table 8: The partitions based on each covering R_i ($1 \leq i \leq l-1, l+1 \leq i \leq m$) and $\mathcal{R}_{(A-\{a_l\})}^{\geq}$, respectively.

U_1	R_1	R_2	.	.	.	R_{l-1}	R_{l+1}	.	.	.	R_m	$\mathcal{R}_{(A-\{a_l\})}^{\geq}$
x_1	P_{1x_1}	P_{2x_1}	.	.	.	$P_{(l-1)x_1}$	$P_{(l+1)x_1}$.	.	.	P_{mx_1}	$P_{(A-\{a_l\})x_1}$
x_2	P_{1x_2}	P_{2x_2}	.	.	.	$P_{(l-1)x_2}$	$P_{(l+1)x_2}$.	.	.	P_{mx_2}	$P_{(A-\{a_l\})x_2}$
.
.
.
x_n	P_{1x_n}	P_{2x_n}	.	.	.	$P_{(l-1)x_n}$	$P_{(l+1)x_n}$.	.	.	P_{mx_n}	$P_{(A-\{a_l\})x_n}$

Table 9: A set-valued information system.

U_1	a_2	a_3	a_4
x_1	$\{0\}$	$\{1, 2\}$	$\{1, 2\}$
x_2	$\{0, 1, 2\}$	$\{1, 2\}$	$\{0, 1, 2\}$
x_3	$\{0, 1\}$	$\{1, 2\}$	$\{1, 2\}$
x_4	$\{0, 2\}$	$\{1, 2\}$	$\{1\}$
x_5	$\{1, 2\}$	$\{1, 2\}$	$\{1\}$
x_6	$\{1, 2\}$	$\{0, 1\}$	$\{0, 1\}$
x_7	$\{0\}$	$\{1, 2\}$	$\{1, 2\}$
x_8	$\{1, 2\}$	$\{0, 1\}$	$\{0, 1\}$

Table 10: The partitions based on R_2, R_3, R_4 and $\mathcal{R}_{(A-\{a_1\})}^{\geq}$, respectively.

U_1	R_2	R_3	R_4	$\mathcal{R}_{(A-\{a_1\})}^{\geq}$
x_1	$\{x_1, x_7\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_7\}$
x_2	$\{x_2, x_3, x_4\}$	$\{U_1\}$	$\{U_1\}$	$\{x_2, x_3, x_4\}$
x_3	$\{x_2, x_3, x_4\}$	$\{U_1\}$	$\{U_1\}$	$\{x_2, x_3, x_4\}$
x_4	$\{x_2, x_3, x_4\}$	$\{U_1\}$	$\{U_1\}$	$\{x_2, x_3, x_4\}$
x_5	$\{x_5, x_6, x_8\}$	$\{U_1\}$	$\{U_1\}$	$\{x_5, x_6, x_8\}$
x_6	$\{x_5, x_6, x_8\}$	$\{U_1\}$	$\{U_1\}$	$\{x_5, x_6, x_8\}$
x_7	$\{x_1, x_7\}$	$\{U_1\}$	$\{U_1\}$	$\{x_1, x_7\}$
x_8	$\{x_5, x_6, x_8\}$	$\{U_1\}$	$\{U_1\}$	$\{x_5, x_6, x_8\}$

information system S_2 based on the compression of S_1 , we get Table 10 by deleting U_1/R_1 based on a_1 . Then we obtain the partition $U_1/\mathcal{R}_{(A-\{a_1\})}^\geq = \{\{x_1, x_7\}, \{x_2, x_3, x_4\}, \{x_5, x_6, x_8\}\}$ and define the mapping $g : U_1 \longrightarrow U_2$ as follows:

$$g(x_1) = g(x_7) = y_1, g(x_2) = g(x_3) = g(x_4) = y_2, g(x_5) = g(x_6) = g(x_8) = y_3,$$

where $U_2 = \{y_1, y_2, y_3\}$. Subsequently, the set-valued information system $(U_1, A - \{a_1\}, V, f_1)$ can be compressed into a relatively small relation system $(U_2, \{g(R_2), g(R_3), g(R_4)\})$. To express clearly, we do not list all the relations in this subsection.

In Example 5.2, we compress the dynamic set-valued information system when deleting an attribute. The same approach can be applied to the set-valued information system when deleting an attribute set.

5.3 Compressing the dynamic set-valued information system when adding an object set

In this subsection, we introduce the equivalence relation for the set-valued information system.

Definition 5.3 Let $S_1 = (U_1, A, V, f_1)$ be a set-valued information system. Then the equivalence relation T_A is defined as

$$T_A = \{(x, y) | \forall a \in A, f(x, a) = f(y, a), x, y \in U_1\}.$$

It is obvious that Pawlak's equivalence relation is the same as that given in Definition 5.3 if the set-valued information system is classical. For the sake of convenience, we denote $[x]_A^1 = \{y | (x, y) \in T_A, x, y \in U_1\}$. There are two steps to compress $S_1 = (U_1, A, V, f_1)$ based on T_A . We first derive the partition $U_1/A = \{C_1, C_2, \dots, C_N\}$ on the basis of T_A . Then we define $g_1(x) = y_k$ for any $x \in C_k$ and obtain $S_2 = (U_2, A, V, f_2)$, where $U_2 = \{y_k | 1 \leq k \leq N\}$, $f_2(y_k, a) = f_1(x, a)$ for $a \in A$, and $x \in g_1^{-1}(y_k)$. Suppose we obtain $S_4 = (U_1 \cup U_3, A, V, f_1 \cup f_2)$ by adding the set-valued information system $S_3 = (U_3, A, V, f_3)$ into S_1 . To compress S_4 by utilizing the compression of the original system S_1 , first, we obtain S_5 by compressing S_3 as S_1 . Second, we compress $S_2 \cup S_5$ as S_1 and get S_7 which is the same as the compression of $S_1 \cup S_3$. To express clearly, the process of the compression of set-valued information systems can be illustrated as follows:

$$\left. \begin{array}{l} S_1 \rightsquigarrow S_2 \\ S_3 \rightsquigarrow S_5 \end{array} \right\} S_6 = S_2 \cup S_5 \rightsquigarrow S_7 \leftarrow S_4 = S_1 \cup S_3 \left\{ \begin{array}{l} S_1 \\ S_3 \end{array} \right.,$$

where \rightsquigarrow (respectively, \leftarrow) denotes the process of the compression of set-valued information systems.

We employ an example to illustrate the data compression of set-valued information systems.

Example 5.4 Table 11 shows the set-valued information system $S_1 = (U_1, A, V, f_1)$. By Definition 5.3, we obtain that $U_1/A = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$. Then we define g_1 and f_2 as follows:

$$g_1(x_1) = g_1(x_2) = y_1, g_1(x_3) = g_1(x_4) = y_2, g_1(x_5) = g_1(x_6) = y_3, f_2(y_i, a_i) = f_1(x, a_i),$$

Table 11: The set-valued information system S_1 .

U_1	a_1	a_2	a_3
x_1	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
x_2	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
x_3	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
x_4	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
x_5	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
x_6	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$

where $x \in g_1^{-1}(y_i)$. Thus we can compress S_1 into $S_2 = (U_2, A, V, f_2)$, where $U_2 = \{g(x)|x \in U_1\}$, and S_2 is shown in Table 12.

Table 12: The compressed set-valued information system S_2 of S_1 .

U_2	a_1	a_2	a_3
y_1	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
y_2	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
y_3	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$

Table 13: The set-valued information system S_3 .

U_3	a_1	a_2	a_3
x_7	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$
x_8	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$
x_9	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
x_{10}	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$

The following example is employed to illustrate how to update the compression when adding an object set.

Example 5.5 By adding S_3 shown in Table 13 into S_1 , we obtain the set-valued information system $S_4 = S_1 \cup S_3$ shown in Table 14. To compress S_4 , as Example 5.4, we compress S_3 to $S_5 = (U_5, A, V, f_5)$ shown in Table 15. Then we compress $S_6 = S_2 \cup S_5$ shown in Table 16 and obtain $S_7 = \{U_7, A, V, f_7\}$ shown in Table 17. Afterwards, we can continue to compress S_7 as Example 4.3 in Section 4.

5.4 Compressing the dynamic set-valued information systems when deleting an object set

Suppose $S_1 = (U_1, A, V, f_1)$ is a set-valued information system, we compress S_1 to $S_2 = (U_2, A, V, f_2)$ under the condition of a homomorphism g_1 . By deleting $S_3 = (U_3, A, V, f_3)$, we obtain $S_4 = (U_4, A, V, f_4)$,

Table 14: The set-valued information system $S_4 = S_1 \cup S_3$.

$U_4 = U_1 \cup U_3$	a_1	a_2	a_3
x_1	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
x_2	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
x_3	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
x_4	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
x_5	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
x_6	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
x_7	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$
x_8	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$
x_9	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
x_{10}	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$

Table 15: The set-valued information system S_5 .

U_5	a_1	a_2	a_3
y_4	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$
y_5	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$

Table 16: The set-valued information system $S_6 = S_2 \cup S_5$.

$U_6 = U_2 \cup U_4$	a_1	a_2	a_3
y_1	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
y_2	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
y_3	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
y_4	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$
y_5	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$

Table 17: The set-valued information system S_7 .

U_7	a_1	a_2	a_3
z_1	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
z_2	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
z_3	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
z_4	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$

where $U_3 \subseteq U_1$ and $U_4 = U_1 - U_3$. There are three steps to compress $S_4 = (U_4, A, V, f_4)$ based on S_2 . By Definition 5.3, we first obtain that $U_1/A = \{[x]_A^1 | x \in U_1\}$ and $U_3/A = \{[x]_A^3 | x \in U_3\}$. It is obvious that $[x]_A^3 \subseteq [x]_A^1$ for any $x \in U_3$. Then we cancel the object $g_1(x)$ in U_2 if $[x]_A^3 = [x]_A^1$ and keep the object $g_1(x)$ in U_2 if $[x]_A^3 \neq [x]_A^1$. Third, we obtain the set-valued information system $S_5 = (U_5, A, V, f_5)$ after the deletion.

Following, we employ an example to illustrate the process of the compression of the updated set-valued information system.

Table 18: The set-valued information system S_3 .

U_3	a_1	a_2	a_3
x_1	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
x_2	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$
x_3	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$

Table 19: The set-valued information system S_4 .

$U_4 = U_1 - U_3$	a_1	a_2	a_3
x_4	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
x_5	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
x_6	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
x_7	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$
x_8	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$
x_9	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
x_{10}	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$

Example 5.6 We take information systems S_4 and S_7 in Example 5.5 as the original set-valued information system S_1 and the compression information system S_2 , respectively. By deleting $S_3 = (U_3, A, V, f)$ shown in Table 18, where $U_3 = \{x_1, x_2, x_3\}$, we obtain the set-valued information system S_4 shown in Table 19. To compress S_4 , we first get that $U_1/A = \{\{x_1, x_2\}, \{x_3, x_4, x_9, x_{10}\}, \{x_5, x_6\}, \{x_7, x_8\}\}$ and $U_3/A = \{\{x_1, x_2\}, \{x_3\}\}$. Obviously, $[x_1]_A^1 = [x_2]_A^1 = \{x_1, x_2\} = [x_1]_A^3 = [x_2]_A^3$ and $[x_3]_A^3 = \{x_3\} \subset \{x_3, x_4, x_9, x_{10}\} = [x_3]_A^1$. Then we cancel z_1 and keep $\{z_2, z_3, z_4\}$ in Table 17. Afterwards, we obtain the compressed set-valued information system S_5 shown in Table 20. We can continue to compress S_5 as Example 4.3 in Section 4.

6 Conclusions

In this paper, we have proposed three tolerance relations for the set-valued information system and studied their basic properties. Then the data compression of set-valued information systems has been

Table 20: The set-valued information system S_5 .

U_5	a_1	a_2	a_3
z_2	$\{0, 1\}$	$\{1\}$	$\{0, 1\}$
z_3	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
z_4	$\{1, 2\}$	$\{0, 2\}$	$\{0, 1\}$

discussed in detail. Afterwards, we have studied the data compression of dynamic set-valued information systems by using the precious compression of the original set-valued information systems.

In the future, we will study the data compression of fuzzy set-valued information systems and dynamic fuzzy set-valued information systems. We will investigate the data compression of interval-valued information systems, fuzzy interval-valued information systems, dynamic interval-valued information systems and dynamic fuzzy interval-valued information systems.

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References

- [1] M. Banerjee, S. K. Pal, Roughness of a fuzzy set, *Information Sciences* 93(3-4) (1996) 235-246.
- [2] R. B. Bhatt, M. Gopal, On the compact computational domain of fuzzy-rough sets, *Pattern Recognition Letters* 26(11) (2005) 1632-1640.
- [3] R. Biswas, On rough sets and fuzzy rough sets, *Bulletin of the Polish Academy of Sciences: Mathematics* 42 (1994) 345-349.
- [4] F. Bobillo, U. Straccia, Generalized fuzzy rough description logics, *Information Sciences* 189 (2012) 43-62.
- [5] A. Capotorti, E. Barbanera, Credit scoring analysis using a fuzzy probabilistic rough set model, *Computational Statistics and Data Analysis* 56(4) (2012) 981-994.
- [6] K. Chakrabarty, R. Biswas, S. Nanda, Fuzziness in rough sets, *Fuzzy Sets and Systems* 110 (2000) 247-251.

- [7] H. M. Chen, T. R. Li, S. J. Qiao, D. Ruan, A rough set based dynamic maintenance approach for approximations in coarsening and refining attribute values, *International Journal of Intelligent Systems* 25(10) (2010) 1005-1026.
- [8] J. H. Dai, W. T. Wang, Q. Xu, H. W. Tian, Uncertainty measurement for interval-valued decision systems based on extended conditional entropy, *Knowledge-Based Systems* 27 (2012) 443-450.
- [9] M. Diker, A. A. Uğur, Textures and covering based rough sets, *Information Sciences* 184(1) (2012) 44-63.
- [10] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, *International Journal of General Systems* 17 (1990) 191-209.
- [11] T. Feng, S. P. Zhang, J. S. Mi, Q. Feng, Reductions of a fuzzy covering decision system, *International Journal of Modelling, Identification and Control* 13(3) (2011) 225-233.
- [12] L. Feng, T. R. Li, D. Ruan, S. R. Gou, A vague-rough set approach for uncertain knowledge acquisition, *Knowledge-Based Systems* 24 (2011) 837-843.
- [13] Z. T. Gong, Z. Y. Xiao, Communicating between information systems based on including degrees, *International Journal of General Systems* 39(2) (2010) 189-206.
- [14] J. W. Grzymala-Busse, W. A. Sedelow Jr., On rough sets and information system homomorphism, *Bulletin of the polish academy of sciences: technical sciences* 36(3) (1988) 233-239.
- [15] Y. Y. Guan, H. K. Wang, Set-valued information systems, *Information Sciences* 176(17) (2006) 2507-2525.
- [16] Q. He, C. X. Wu, D. G. Chen, Fuzzy rough set based attribute reduction for information systems with fuzzy decisions, *Knowledge-Based Systems* 24(5) (2011) 689-696.
- [17] R. Jensen, Q. Shen, Semantics-preserving dimensionality reduction: rough and fuzzy-rough-based approaches, *IEEE Transactions on Knowledge and Data Engineering* 16(12) (2004) 1457-1471.
- [18] D. Y. Li, Y. C. Ma, Invariant characters of information systems under some homomorphisms, *Information Sciences* 129(1-4) (2000) 211-220.
- [19] T. R. Li, D. Ruan, W. Geert, J. Song, Y. Xu, A rough sets based characteristic relation approach for dynamic attribute generalization in data mining, *Knowledge-Based Systems* 20(5) (2007) 485-494.

- [20] G. L. Liu, The axiomatization of the rough set upper approximation operations, *Fundamenta Informaticae* 69(3) (2006) 331-342.
- [21] G. L. Liu, Rough set theory based on two universal sets and its applications, *Knowledge-Based Systems* 23(2) (2010) 110-115.
- [22] D. Liu, T. R. Li, D. Ruan, J. B. Zhang, Incremental learning optimization on knowledge discovery in dynamic business intelligent systems, *Journal of Global Optimization* 51(2) (2011) 325-344.
- [23] D. Liu, T. R. Li, D. Ruan, W. L. Zou, An incremental approach for inducing knowledge from dynamic information systems, *Fundamenta Informaticae* 94(2) (2009) 245-260.
- [24] N. N. Morsi, M. M. Yakout, Axiomatics for fuzzy rough sets, *Fuzzy Sets and Systems* 100(1-3) (1998) 327-342.
- [25] S. Nanda, S. Majumdar, Fuzzy rough sets, *Fuzzy Sets and Systems* 45(2) (1992) 157-160.
- [26] Z. Pawlak, Rough sets, *International Journal of Computer and Information Sciences* 11(5) (1982) 341-356.
- [27] Z. Pawlak, A. Skowron, Rudiments of rough sets, *Information Sciences* 177 (2007) 3-27.
- [28] Z. Pawlak, A. Skowron, Rough sets: Some extensions, *Information Sciences* 177 (2007) 28-40.
- [29] Z. Pawlak, A. Skowron, Rough sets and boolean reasoning, *Information Sciences* 177 (2007) 41-73.
- [30] Y. H. Qian, C. Y. Dang, J. Y. Liang, D. W. Tang, Set-valued ordered information systems, *Information Sciences* 179 (2009) 2809-2832.
- [31] A. Skowron, The rough set theory and evidence theory, *Fundamenta Informaticae* 13 (1990) 245-262.
- [32] D. Ślęzak, W. Ziarko, The investigation of the Bayesian rough set model, *International Journal of Approximate Reasoning* 40(1-2) (2005) 81-91.
- [33] C. Z. Wang, C. X. Wu, D. G. Chen, A systematic study on attribute reduction with rough sets based on general binary relations, *Information Sciences* 178(9) (2008) 2237-2261.
- [34] S. P. Wang, Q. X. Zhu, W. Zhu, F. Min, Matroidal structure of rough sets and its characterization to attribute reduction, *Knowledge-Based Systems* (2012) <http://dx.doi.org/10.1016/j.knosys.2012.06.006>.

- [35] T. Yang, Q. G. Li, Reduction about approximation spaces of covering generalized rough sets, *International Journal of Approximate Reasoning* 51(3) (2010) 335-345.
- [36] X. B. Yang, M. Zhang, H. L. Dou, Neighborhood systems-based rough sets in incomplete information system, *Knowledge-Based Systems* 24(6) (2011) 858-867.
- [37] Y. Y. Yao, Probabilistic approaches to rough sets, *Expert Systems* 20(5) (2003) 287-297.
- [38] Y. Y. Yao, Three-way decisions with probabilistic rough sets, *Information Sciences* 180(3) (2010) 341-353.
- [39] Y. Y. Yao, Y. Zhao, Attribute reduction in decision-theoretic rough set models, *Information Sciences* 178(17) (2008) 3356-3373.
- [40] W. Zakowski, Approximations in the space (u, π) , *Demonstratio Mathematica* 16 (1983) 761-769.
- [41] J. B. Zhang, T. R. Li, D. Ruan, D. Liu, Rough sets based matrix approaches with dynamic attribute variation in set-valued information systems, *International Journal of Approximate Reasoning* 53(4) (2012) 620-635.
- [42] W. Zhu, Topological approaches to covering rough sets, *Information Sciences* 177(6) (2007) 1499-1508.
- [43] P. Zhu, Covering rough sets based on neighborhoods: An approach without using neighborhoods, *International Journal of Approximate Reasoning* 52(3) (2011) 461-472.
- [44] P. Zhu, Q. Y. Wen, Some improved results on communication between information systems, *Information Sciences* 180(18) (2010) 3521-3531.
- [45] W. Ziarko, Probabilistic approach to rough sets, *International Journal of Approximate Reasoning* 49(2) (2008) 272-284.